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SPATIAL SIMPLE WAVES ON A SHEAR FLOW

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The paper studies simple waves of the shallow-water equations describing three-dimensional wave motions of a rotational liquid in a free-boundary layer. Simple wave equations are derived for the general case. The existence of unsteady or steady simple waves adjacent continuously to a given steady shear flow along a characteristic surface is proved. Exact solutions of the equations describing steady simple waves were found. These solutions can be treated as extension of Prandtl–Mayer waves for sheared flows. For shearless flows, a general solution of the system of equations describing unsteady spatial simple waves was found.

1. Formulation of the Problem. The paper considers the equations

$$\begin{gathered} u_t + (\boldsymbol{U}\cdot\nabla)u + p_x/\rho = 0, \qquad v_t + (\boldsymbol{U}\cdot\nabla)v + p_y/\rho = 0, \\ p_z/\rho = -g, \qquad \operatorname{div}\boldsymbol{U} = 0 \end{gathered} \tag{1.1}$$

describing unsteady three-dimensional motions of an ideal incompressible fluid in a long-wave approximation. Model (1.1) is the long-wave limit $\varepsilon = H_0/L_0 \to 0$ of the exact Euler equations (H_0 is the characteristic vertical scale and L_0 is the characteristic horizontal scale). Here $\boldsymbol{U} = (u, v, w)$ is the fluid velocity, p is the pressure, g is the acceleration of gravity, $\rho = \text{const}$ is the fluid density, x, y, and z are the Cartesian coordinates in space, and t is time. The present paper focuses on the free-boundary problem for system (1.1) that describes wave motions of a fluid in the layer $0 \leqslant z \leqslant h(t, x, y)$, where h is the fluid-layer depth. Let us formulate boundary conditions of the problem. On the even bottom $z = 0$, the boundary condition $w = 0$ is specified, and the dynamic condition $p = p_0 = \text{const}$ is imposed on the free surface. The kinematic condition on the free boundary can be written as

$$h_t + \operatorname{div}\Big(\int\limits_0^h \boldsymbol{u}\,dz\Big) = 0,$$

where $\boldsymbol{u} = (u, v)$ is the projection of the velocity onto a plane orthogonal to the z axis. Using the third equation of the system, we derive the hydrostatic distribution of pressure over the depth:

$$p = p_0 + \rho g(h - z).$$

In the approximate theory considered, the curl vector $\boldsymbol{\Omega} = (-v_z, u_z, v_x - u_y)$ satisfies the equation

$$\boldsymbol{\Omega}_t + (\boldsymbol{U}\cdot\nabla)\boldsymbol{\Omega} = (\boldsymbol{\Omega}\cdot\nabla)\boldsymbol{U}.$$

Equations for the first two components of the vector $\boldsymbol{\Omega}$ are written as

$$-v_{zt} - (\boldsymbol{U}\cdot\nabla)v_z + v_z u_x - u_z v_x = 0, \quad u_{zt} - (\boldsymbol{U}\cdot\nabla)u_z + v_z u_y - u_z v_y = 0. \tag{1.2}$$

From (1.2) it follows that if the equalities $u_z = 0$ and $v_z = 0$ hold for $t = 0$, then $u_z = 0$ and $v_z = 0$ for $t > 0$. In the last case, system (1.1) reduces to the classical shallow-water equations

$$u_t + (\boldsymbol{u}\cdot\nabla)u + gh_x = 0, \qquad v_t + (\boldsymbol{u}\cdot\nabla)v + gh_x = 0,$$

$$h_t + \operatorname{div}(h\boldsymbol{u}) = 0, \qquad w = -z(u_x + v_y),$$

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which describe flows free of a vertical shear of velocity (u and v do not depend on the variable z). In the general case, however, the vertical profiles of velocity can be rather arbitrary. Below, by shear flow is meant the class of solutions of Eqs. (1.1) characterized by the inequality $u_z^2 + v_z^2 \neq 0$. By uniform shear flow is meant a particular solution of (1.1) with the functions independent of x, y, and t:

$$u = u_0(z), \qquad v = v_0(z), \qquad w = 0, \qquad p = -\rho g z + \text{const}.$$

In analysis of shear flows, it is convenient to convert Eqs. (1.1) to the Eulerian–Lagrangian variables t', x', y', and λ. The change of variables

$$x = x', \qquad y = y', \qquad t = t', \qquad z = \Phi(t', x', y', \lambda)$$

is performed using the function Φ (which is a solution of the Cauchy problem):

$$\Phi_t + u(t, x, y, \Phi)\Phi_x + v(t, x, y, \Phi)\Phi_y = w(t, x, y, \Phi), \qquad \Phi|_{t=0} = \Phi_0(x, y, \lambda).$$

The initial data $\Phi_0(x, y, \lambda)$ are chosen so as to satisfy the conditions $\Phi_0(x, y, 0) = 0$ and $\Phi_0(x, y, 1) = h_0(x, y) = h(0, x, y)$. Here λ is a Lagrangian coordinate that marks the material surfaces. From the above equation, it follows that $\Phi(t, x, y, 0) = 0$ and $\Phi(t, x, y, 1) = h(t, x, y)$. Therefore, in the new variables, the region occupied by the fluid transforms into the fixed layer $0 \leqslant \lambda \leqslant 1$.

In the Eulerian–Lagrangian variables, the equations governing fluid motion take the form

$$u_t + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} + g\nabla h = 0, \qquad H_t + \operatorname{div}(H\boldsymbol{u}) = 0. \tag{1.3}$$

Here $H = \Phi_\lambda \neq 0$ is the Jacobian of the conversion to the new variables (the primes of the independent variables are omitted); the operations ∇ and div are performed with respect to the variables x and y. The functions Φ, w, and h are expressed in terms of the vector $\boldsymbol{u}$ and Jacobian H by the relations

$$\Phi = \int_0^\lambda H\, d\nu, \qquad w = \Phi_t + u\Phi_x + v\Phi_y, \qquad h = \int_0^1 H\, d\lambda.$$

In the new variables, the shear flows are characterized by the inequality $u_\lambda^2 + v_\lambda^2 \neq 0$.

Particular solutions of the form $\boldsymbol{u} = \boldsymbol{u}(\alpha, \lambda)$ and $H = H(\alpha, \lambda)$, where $\alpha = \alpha(t, x, y)$, are called the simple waves of system (1.5). In the Eulerian coordinates, they correspond to the solutions of system (1.1) that satisfy the equalities

$$\boldsymbol{u} = \boldsymbol{u}(\alpha, z), \qquad h = h(\alpha).$$

In this class of solutions, the vertical velocity component is given by

$$w = -\int_0^z \boldsymbol{u}_\alpha(\alpha, z')dz' \cdot \nabla\alpha.$$

From (1.3), we obtain the simple wave equations

$$\boldsymbol{u}_\alpha \frac{d\alpha}{dt} + gh_\alpha \nabla\alpha = 0, \qquad H_\alpha \frac{d\alpha}{dt} + H(\boldsymbol{u}_\alpha \cdot \nabla\alpha) = 0. \tag{1.4}$$

Here $d\alpha/dt = \alpha_t + (\boldsymbol{u} \cdot \nabla)\alpha$. Below, we prove the existence of spatial simple waves propagating over an arbitrary uniform shear flow, derive new exact solutions in this class, and construct a general solution of the simple-wave equations in the class of shearless flows. We note that previous studies were concerned primarily with plane–parallel flows. Thus, Teshukov [1] and Liapidevskii and Teshukov [2] studied the problem of the existence of solutions for plane–parallel flows of vortical shallow water. A number of exact solutions describing the propagation of simple waves over plane–parallel shear flow were derived in [1, 3–9]. Some properties of spatial simple waves for gas-dynamic equations were analyzed by Ovsyannikov [10].

2. Transformation of Simple-Wave Equations. We multiply the first equation in system (1.4) by $\boldsymbol{u}_\alpha$ and then, using the second equation, eliminate $\boldsymbol{u}_\alpha \cdot \nabla\alpha$. Assuming that $d\alpha/dt \neq 0$, we obtain the equality

$$(\boldsymbol{u}_\alpha)^2 = gh_\alpha H_\alpha / H. \tag{2.1}$$

Using Eqs. (1.4), we find the derivatives of the function α:

$$\nabla\alpha = -\frac{1}{gh_\alpha}\,\frac{d\alpha}{dt}\,\boldsymbol{u}_\alpha = L\boldsymbol{u}_\alpha,$$

$$\alpha_t = -\boldsymbol{u}\cdot\nabla\alpha - (H_\alpha/H)\boldsymbol{u}_\alpha\cdot\nabla\alpha = -L(\boldsymbol{u}\cdot\boldsymbol{u}_\alpha + gh_\alpha)$$

and establish that the ratio

$$\boldsymbol{u}_\alpha/(\boldsymbol{u}\cdot\boldsymbol{u}_\alpha + gh_\alpha) = -\nabla\alpha/\alpha_t \tag{2.2}$$

does not depend on λ [the ratio on the right side of equality (2.2) is a function of the variables t, x, and y]. Hence, we can write

$$\boldsymbol{u}_\alpha/(\boldsymbol{u}\cdot\boldsymbol{u}_\alpha + gh_\alpha) = \boldsymbol{n}(\alpha)/k(\alpha), \tag{2.3}$$

where $\boldsymbol{n}(\alpha)$ is a specified unit vector and $k(\alpha)$ is the unknown function. It follows from (2.2) that the vector $\boldsymbol{n}$ is orthogonal to the surface $\alpha = \text{const}$, and k is the normal velocity of motion of the surface $\alpha = \text{const}$. Using the consequence of Eqs. (2.3)

$$\boldsymbol{u}\cdot\boldsymbol{u}_\alpha = u_n(\boldsymbol{u}\cdot\boldsymbol{u}_\alpha + gh_\alpha)/k,$$

we reduce Eqs. (2.1) and (2.3) to the form

$$\boldsymbol{u}_\alpha = -gh_\alpha\boldsymbol{n}/(u_n - k), \qquad H_\alpha = gh_\alpha H/(u_n - k)^2. \tag{2.4}$$

Here $u_n = \boldsymbol{u}\cdot\boldsymbol{n}$. We note that Teshukov [11] (see also Liapidevskii and Teshukov [2]) proposed an extension of the theory of characteristics for systems with operator coefficients that makes it possible, in particular, to find the characteristic surfaces of the integrodifferential equations (1.3). Integration of the last equation over the variable λ yields the characteristic equation

$$1 = g\int_0^1 \frac{H\,d\lambda}{(u_n - k)^2}, \tag{2.5}$$

from which it follows that each surface $\alpha = \text{const}$ is a characteristic surface of system (1.3). Analyzing the behavior of the function of k on the right side of equality (2.5), we readily establish the existence of two real roots of the characteristic equation (k_1 and k_2) that satisfy the inequalities $k_1 < \min u_n$ and $k_2 > \max u_n$ (here the minimum and maximum of the function u_n are evaluated with respect to the variable λ). Therefore, below we consider the simple waves corresponding to both the first an second characteristic families. Equations (2.4) are similar in form to the simple-wave equations for plane–parallel shear flow [1]. If we set $\boldsymbol{n} = \boldsymbol{n}_0 = \text{const}$ and $\boldsymbol{u} = u\boldsymbol{n}_0$, Eqs. (2.4) and (2.5) reduce to the simple-wave equations for plane–parallel flow. Generally, in the spatial case, it is convenient to introduce the normal (u_n) and tangent (u_τ) velocity vectors to the simple-wave front:

$$\boldsymbol{u} = u_n\boldsymbol{n} + u_\tau\boldsymbol{\tau}. \tag{2.6}$$

Here $\boldsymbol{n} = (\cos\beta, \sin\beta)$, $\boldsymbol{\tau} = (-\sin\beta, \cos\beta)$, and $\beta = \beta(\alpha)$ is a specified function.

Differentiation of (2.6) with respect to the variable α yields the following representation of the derivative in the basis $\boldsymbol{n}$ and $\boldsymbol{\tau}$:

$$\boldsymbol{u}_\alpha = ((u_n)_\alpha - (\beta')u_\tau)\boldsymbol{n} + ((u_\tau)_\alpha + \beta' u_n)\boldsymbol{\tau}. \tag{2.7}$$

Using (2.7), we write Eqs. (2.4) and (2.5) in the form

$$(u_n)_\alpha + \frac{gh_\alpha}{u_n - k} - \beta' u_\tau = 0, \quad (u_\tau)_\alpha + \beta' u_n = 0, \quad H_\alpha = \frac{gh_\alpha H}{(u_n - k)^2}. \tag{2.8}$$

Combination of the first two equations in system (2.8) yields the following second-order equation for the unknown u_n:

$$\Big(\frac{1}{\beta'}\Big((u_n)_\alpha + \frac{gh_\alpha}{u_n - k}\Big)\Big)_\alpha + \beta' u_n = 0. \tag{2.9}$$

The integrodifferential equations (2.5) and (2.8) form a closed system for determining the velocity $\boldsymbol{u}$ and the quantity H as functions of the variables α and λ. Next, we write the equations defining the function $\alpha(t, x, y)$. The following equality is valid on the surface $\alpha = \text{const}$:

$$\alpha_t dt + \nabla\alpha\cdot d\boldsymbol{x} = 0.$$

From this, by virtue of (2.2) and (2.3), we obtain the relation

$$\boldsymbol{n}(\alpha)\cdot d\boldsymbol{x}-k(\alpha)\,dt=0. \tag{2.10}$$

Since the coefficients of the differential form are constant on the surface $\alpha=\text{const}$, we integrate equality (2.10):

$$\boldsymbol{n}(\alpha)\cdot\boldsymbol{x}-k(\alpha)t=m(\alpha). \tag{2.11}$$

Here $m(\alpha)$ is an arbitrary function. The function $\alpha(t,x,y)$ can be found from Eq. (2.11) for $m(\alpha)$ fixed. The following equation can be used instead of (2.11):

$$\boldsymbol{u}_\alpha\cdot\boldsymbol{x}/(\boldsymbol{u}\cdot\boldsymbol{u}_\alpha+gh_\alpha)-t=m(\alpha). \tag{2.12}$$

Let us show that if we specify the arbitrary functions $\beta(\alpha)$ and $m(\alpha)$, integrate Eqs. (2.8), and find $\alpha(t,x,y)$ from (2.11), then, the functions $\boldsymbol{u}$, H, and α obtained will satisfy Eqs. (1.4). Here we assume that the inequality $\boldsymbol{n}_\alpha\cdot\boldsymbol{x}-k'(\alpha)t-m'(\alpha)\neq 0$ holds. Using this condition, we apply the implicit-function theorem to Eq. (2.11) and find $\alpha(t,x,y)$ locally.

Differentiation of (2.11) with respect to the variables $\boldsymbol{x}$ and t yields

$$(\boldsymbol{n}_\alpha\cdot\boldsymbol{x}-k'(\alpha)t-m'(\alpha))\nabla\alpha+\boldsymbol{n}=0,\quad(\boldsymbol{n}_\alpha\cdot\boldsymbol{x}-k'(\alpha)t-m'(\alpha))\alpha_t-k=0. \tag{2.13}$$

Using Eqs. (2.8) and their consequences (2.4), from (2.13) we obtain the relations

$$\nabla\alpha=L_1\boldsymbol{u}_\alpha,\qquad\frac{d\alpha}{dt}=\alpha_t+(\boldsymbol{u}\cdot\nabla)\alpha=-gL_1h_\alpha, \tag{2.14}$$

where $L_1=-k(\boldsymbol{n}_\alpha\cdot\boldsymbol{x}-k'(\alpha)t-m'(\alpha))^{-1}(\boldsymbol{u}\cdot\boldsymbol{u}_\alpha+gh_\alpha)^{-1}$. Obviously, (2.14) implies the equalities

$$\boldsymbol{u}_\alpha\frac{d\alpha}{dt}+gh_\alpha\nabla\alpha=0,\qquad H_\alpha\frac{d\alpha}{dt}+H\boldsymbol{u}_\alpha\cdot\nabla\alpha=L_1(-gH_\alpha h_\alpha+\boldsymbol{u}_\alpha^2)=0.$$

Thus, $\boldsymbol{u}$, H, and α satisfy (1.4). Relation (2.14) implies the potentiality of the vector $\boldsymbol{u}$:

$$\boldsymbol{u}(\alpha,\lambda)=\nabla\phi(x,y,\lambda) \tag{2.15}$$

(the potential exists because $u_\alpha\alpha_y-v_\alpha\alpha_x=0$). The potential ϕ is defined by the formula

$$\phi=\boldsymbol{x}\cdot\boldsymbol{u}-\Big(\frac{|\boldsymbol{u}|^2}{2}+gh\Big)t-\int_{\alpha_0}^{\alpha}(\boldsymbol{u}\cdot\boldsymbol{u}_\alpha+gh_\alpha)(\alpha',\lambda)m(\alpha')\,d\alpha'. \tag{2.16}$$

Indeed, relation (2.15) is a consequence of relation (2.12) and equality (2.16) differentiated with respect to $\boldsymbol{x}$. Differentiating (2.16) with respect to t and using (2.12), we obtain an analogue of the Cauchy–Lagrange integral:

$$\phi_t=-|\boldsymbol{u}|^2/2-gh.$$

In Eulerian variables, the vector $\boldsymbol{u}$ is not potential. Upon change of the variables, the equality $u_y(\alpha,\lambda)-v_x(\alpha,\lambda)=0$ reduces to

$$\boldsymbol{\Omega}\cdot\boldsymbol{\nu}=0,$$

where $\boldsymbol{\nu}$ is a normal vector to the surface $\lambda=\text{const}$ in the three-dimensional space (x,y,z). Hence, in Eulerian variables, a simple wave is characterized by vanishing of the normal component of the curl on the above-mentioned family of material surfaces. By virtue of the Helmholtz theorems, the normal vortex component remains zero during flow evolution if it vanishes at $t=0$.

Below, instead of the finite relation (2.5), it is more convenient to use the equivalent differential equation derived by differentiating (2.5) with respect to α and using Eqs. (2.8). As a result, we have

$$k_\alpha=-\bigg(2\int_0^1\frac{H\,d\lambda}{(u_n-k)^3}\bigg)^{-1}\bigg(3gh_\alpha\int_0^1\frac{H\,d\lambda}{(u_n-k)^4}+2\beta'\int_0^1\frac{Hu_\tau\,d\lambda}{(u_n-k)^3}\bigg). \tag{2.17}$$

If the Cauchy data for (2.17) are chosen such that (2.5) is satisfied on the initial surface, equality (2.5) is a consequence of (2.8) and (2.17) for all α.

As a result, construction of a simple wave reduces to integration of the system of integrodifferential equations (2.8) and (2.17). If the functions u_n, u_τ, H, and k are known, the vector $\boldsymbol{u}$ is defined by formula (2.6), and $\alpha(t,x,y)$ can be found from Eq. (2.11).

3. Simple Waves on Spatial Shearless Flow. The simple-wave equations are easily integrated in the class of flows without a vertical shear, defined by the equalities $u_\lambda = v_\lambda = 0$. In this case, the characteristic equation (2.5) reduces to the form

$$(u_n - k)^2 = gh,$$

and $u_n - k = \pm\sqrt{gh}$ in the domain of definition of a simple wave. In this section, we assume that $\alpha(t, x, y) = h(t, x, y)$. Integration of Eq. (2.4) yields

$$\boldsymbol{u}(h) = \mp \int\limits_{h_0}^{h} \sqrt{g/h'}(\cos\beta(h'), \sin\beta(h'))\, dh' + \boldsymbol{u}_0.$$

Here $h_0 = \text{const}$ and $\boldsymbol{u}_0$ is an arbitrary constant vector. A similar relation is obtained by integrating Eq. (2.9). We assume that the function $\beta = \beta(h)$ is monotonic, and, hence, can be inverted: $h = h(\beta)$. With allowance for this, Eq. (2.9) is written as

$$(u_n \pm 2\sqrt{gh})_{\beta\beta} + (u_n \pm 2\sqrt{gh}) = \pm 2\sqrt{gh}. \qquad (3.1)$$

We note that after the replacement $u_n \to u$, the quantities $u_n \pm 2\sqrt{gh}$ coincide with the Riemann invariants used in studies of plane–parallel flows. The general solution of Eq. (3.1) has the form

$$u_n \pm 2\sqrt{gh} = \left(A_0 \pm 2\int\limits_{\beta_0}^{\beta} \sqrt{gh(\beta')}\cos\beta'\, d\beta'\right)\sin\beta + \left(B_0 \mp 2\int\limits_{\beta_0}^{\beta} \sqrt{gh(\beta')}\sin\beta'\, d\beta'\right)\cos\beta, \qquad (3.2)$$

where A_0 and B_0 are arbitrary constants. From (2.8), we obtain

$$u_\tau = (u_n \pm 2\sqrt{gh})_\beta = \left(A_0 \pm 2\int\limits_{\beta_0}^{\beta} \sqrt{gh(\beta')}\cos\beta'\, d\beta'\right)\cos\beta$$
$$- \left(B_0 \mp 2\int\limits_{\beta_0}^{\beta} \sqrt{gh(\beta')}\sin\beta'\, d\beta'\right)\sin\beta. \qquad (3.3)$$

Formulas (3.2) and (3.3) define completely (with appropriate choice of arbitrary constants) the velocity vector

$$\boldsymbol{u}(h) = (u_n\cos\beta - u_\tau\sin\beta, u_n\sin\beta + u_\tau\cos\beta)$$

$$= \mp 2\sqrt{gh}(\cos\beta(h), \sin\beta(h)) \mp \left(\int\limits_{\beta_0}^{\beta} \sqrt{gh(\beta')}\sin\beta'\, d\beta', -\int\limits_{\beta_0}^{\beta} \sqrt{gh(\beta')}\cos\beta'\, d\beta'\right) + (B_0, A_0)$$

in a simple wave adjacent to a specified constant flow of depth $h = h_0$ moving with velocity $\boldsymbol{u}(h_0)$ (on the contact surface $\beta = \beta_0$). The function h is obtained from the equation

$$\cos\beta(h)x + \sin\beta(h)y - (u_n \pm \sqrt{gh})t = m(h),$$

where $m(h)$ is an arbitrary function. We note that relations (2.2) and (2.3) imply the following differential equation for the function $h(t, x, y)$:

$$h_t + (u_n \pm \sqrt{gh})(\cos\beta h_x + \sin\beta h_y) = 0. \qquad (3.4)$$

Therefore, to find h, it is necessary to solve the Cauchy problem

$$h\Big|_{t=0} = h_0(x, y)$$

for Eq. (3.4) with special initial data. The initial function $h_0(x, y)$ must be a solution of the equation

$$-\sin\beta(h_0)h_{0x} + \cos\beta(h_0)h_{0y} = 0.$$

With such a choice, the level lines $h_0(x, y)$ are straight lines and the level surfaces $h(t, x, y)$ are planes.

As a result, it is shown that the general solution of the equations of spatial simple waves propagating over a flow without a velocity shear depends on the two arbitrary functions of the same argument: $\beta(h)$ and $m(h)$.

4. Existence of an Unsteady Simple Wave Propagating over the Shear Flow. We choose the function $\alpha(t,x,y) = h(t,x,y)$. Let us consider the Cauchy problem

$$(u_n)_h = -g/(u_n - k) + \beta' u_\tau, \qquad (u_\tau)_h = -\beta' u_n, \qquad H_h = gH/(u_n - k)^2,$$

$$k_h = -\left(2\int_0^1 \frac{H\,d\lambda}{(u_n - k)^3}\right)^{-1}\left(3g\int_0^1 \frac{H\,d\lambda}{(u_n - k)^4} + 2\beta'\int_0^1 \frac{Hu_\tau\,d\lambda}{(u_n - k)^3}\right), \tag{4.1}$$

$$u_n\Big|_{h=h_0} = \boldsymbol{u}_0(\lambda)\boldsymbol{n}(h_0), \quad u_\tau\Big|_{h=h_0} = \boldsymbol{u}_0(\lambda)\boldsymbol{\tau}(h_0), \quad H\Big|_{h=h_0} = H_0(\lambda), \quad k\Big|_{h=h_0} = k_0.$$

Here $\boldsymbol{u}_0(\lambda)$ and $H_0(\lambda)$ are given functions and k_0 is a root of Eq. (2.5) calculated for $h = h_0$ using the Cauchy data. The functions $\beta(h)$ and, hence, $\boldsymbol{n}(h) = (\cos\beta(h), \sin\beta(h))$ and $\boldsymbol{\tau}(h) = (-\sin\beta(h), \cos\beta(h))$ are given. It is assumed that the function $\beta(h)$ is twice differentiable. The Cauchy data in (4.1) provide for continuous joining of the simple wave and the given uniform shear flow with constant depth $h = h_0$ and velocity $\boldsymbol{u}_0(\lambda)$.

Let us prove the existence of a solution of the problem (4.1), assuming that the functions $\boldsymbol{u}_0(\lambda)$ and $H_0(\lambda)$ are continuously differentiable if $\lambda \in [0,1]$, $|\boldsymbol{u}_0(\lambda)\cdot\boldsymbol{n}(h_0) - k_0| > \delta > 0$, and $|H_0| > \delta > 0$ (δ is a constant). The simple-wave equations (4.1) are integrodifferential; therefore, to prove the existence of a solution of the Cauchy problem, we use the following theorem for differential equations in Banach space [12].

The Cauchy problem

$$\frac{d\boldsymbol{V}}{dh} = \boldsymbol{F}(\boldsymbol{V}), \qquad \boldsymbol{V}\Big|_{h=h_0} = \boldsymbol{V}_0 \tag{4.2}$$

has a unique solution defined for $|h - h_0| < \min(\varepsilon M^{-1}, K^{-1})$, which belongs to a ball $\|\boldsymbol{V} - \boldsymbol{V}_0\| < \varepsilon$ of a Banach space B if the nonlinear operator $\boldsymbol{F}$ on this ball satisfies the inequalities

$$\|\boldsymbol{F}(\boldsymbol{V})\| < M, \qquad \|\boldsymbol{F}(\boldsymbol{V}_1) - \boldsymbol{F}(\boldsymbol{V}_2)\| \leqslant K\|\boldsymbol{V}_1 - \boldsymbol{V}_2\|. \tag{4.3}$$

Since we seek solutions differentiable with respect to λ, let us extend Eqs. (4.1) to the derivatives $u_{n\lambda}$ and $u_{\tau\lambda}$:

$$(u_{n\lambda})_h = gu_{n\lambda}/((u_n - k)^2) + \beta' u_{\tau\lambda}, \qquad (u_{\tau\lambda})_h = -\beta' u_{n\lambda},$$
$$u_{n\lambda}\Big|_{h=h_0} = \boldsymbol{u}_0'(\lambda)\boldsymbol{n}(h_0), \qquad u_{\tau\lambda}\Big|_{h=h_0} = \boldsymbol{u}_0'(\lambda)\boldsymbol{\tau}(h_0). \tag{4.4}$$

We introduce a Banach surface B of elements $\boldsymbol{V} = (u_n, u_{n\lambda}, u_\tau, u_{\tau\lambda}, H, k)$ with a norm

$$\|\boldsymbol{V}\| = \max_\lambda |u_n| + \max_\lambda |u_{n\lambda}| + \max_\lambda |u_\tau| + \max_\lambda |u_{\tau\lambda}| + \max_\lambda |H| + |k|,$$

where $\lambda \in [0,1]$. Let $\boldsymbol{V}_0 \in B$ be the vector of the initial data of the problem (4.1), (4.4). We consider a ball $\|\boldsymbol{V} - \boldsymbol{V}_0\| < \delta/2$. For the elements of this ball, $|u_n - k| > \delta/2$ and $|H| > \delta/2$. Indeed,

$$|u_n - k| = |u_{n0} - k_0 + u_n - u_{n0} + k_0 - k| \geqslant |u_{n0} - k_0| - \|\boldsymbol{V} - \boldsymbol{V}_0\| > \delta/2,$$

$$|H| = |H_0 + H - H_0| \geqslant |H_0| - \|\boldsymbol{V} - \boldsymbol{V}_0\| > \delta/2.$$

Taking the above inequalities into account, it is easy to show that the right sides of Eqs. (4.1) and (4.4) satisfy inequalities of the form of (4.3) with some $M = M(\delta)$ and $K = K(\delta)$. Then, by virtue of the theorem formulated above, the problem (4.1), (4.4) has a unique solution for $|h - h_0| < \delta_1(\delta)$.

To complete the construction of a simple wave, it is necessary to find the function $h(t,x,y)$ from the equation

$$\boldsymbol{n}(h)\boldsymbol{x} - k(h)t = m(h). \tag{4.5}$$

It follows from (4.5) that the level surfaces $h = \text{const}$ are planes in the space (x,y,t). Since these planes are generally nonparallel, it is impossible to determine h unambiguously at points of intersection of the planes. Therefore, a simple-wave solution can be constructed only locally in subdomains of the space (x,y,t) that do not contain points of intersection of the above-mentioned planes.

5. Simple Waves on a Steady Shear Flow. Let us consider a simple wave on a steady shear flow. In this case, Eq. (2.1) also holds for $\alpha = \alpha(x, y)$. Using the relations

$$\nabla\alpha = -\boldsymbol{u}\cdot\nabla\alpha\boldsymbol{u}_\alpha/(gh_\alpha) = L\boldsymbol{u}_\alpha,$$

$$0 = -\boldsymbol{u}\cdot\nabla\alpha - (H/H_\alpha)\boldsymbol{u}_\alpha\cdot\nabla\alpha = -L(\boldsymbol{u}\cdot\boldsymbol{u}_\alpha + gh_\alpha)$$

(as in Sec. 2), we obtain the differential equations

$$\boldsymbol{u}_\alpha = -gh_\alpha\boldsymbol{n}/u_n, \qquad H_\alpha = gh_\alpha H/u_n^2, \tag{5.1}$$

where $u_n = \boldsymbol{u}\cdot\boldsymbol{n}$, $\boldsymbol{n} = (-\sin\gamma, \cos\gamma)$, and $\gamma = \gamma(\alpha)$ is the unknown function (it was a given function in the unsteady case). This function can be found using the characteristic equation

$$1 = g\int\limits_0^1 \frac{H\,d\lambda}{u_n^2}. \tag{5.2}$$

By the definition, $\gamma(\alpha)$ specifies the angle between the x axis and the direction of the characteristic $\alpha = \text{const}$ in the plane (x, y). Below, we specify which of the two possible values of the angle is denoted by $\gamma(\alpha)$. Let us introduce polar coordinates q and θ on a hodograph plane using the relation $\boldsymbol{u} = q(\cos\theta, \sin\theta)$ and setting $\alpha(x, y) = h(x, y)$. In the new variables, we write Eqs. (5.1) as

$$qq_h + g = 0, \qquad \theta_h = -g\cot\,(\theta-\gamma)/q^2, \qquad H_h = gH/(q^2\sin^2(\theta-\gamma)) \tag{5.3}$$

and the characteristic equation (5.2) as

$$\chi(\gamma) = 1 - g\int\limits_0^1 \frac{H\,d\lambda}{q^2\sin^2(\theta-\gamma)} = 0. \tag{5.4}$$

It follows from (5.4) that $\chi(\gamma)$ is a periodic function: $\chi(\gamma) = \chi(\gamma+\pi)$. We note that the angles γ and $\gamma+\pi$, differing in sign, correspond to normals to the same characteristic surface. Therefore, in what follows, we shall seek only such roots γ of the characteristic equation that differ in the modulus π. The function χ is defined on the subset of the real axis consisting of a complement to the segments $[\theta_{\min}(h)+l\pi, \theta_{\max}(h)+l\pi]$, where l is an arbitrary integer [the segments are obtained by shifting the value area of the function $\theta(h,\lambda)$ by $l\pi$ for fixed h and $0 \leqslant \lambda \leqslant 1$]. The domain of definition is not empty if $\theta_{\max}(h) - \theta_{\min}(h) < \pi$. In the last case, we choose the angle γ such that the inequality $|(\theta_{\max}+\theta_{\min})/2 - \gamma| < \pi/2$ is satisfied. With allowance for this, it will suffice to study the behavior of the function $\chi(\gamma)$ on the segments $[\theta_{\max}-\pi, \theta_{\min}]$ and $[\theta_{\max}, \theta_{\min}+\pi]$. This continuous function is finite at the interior points of the indicated segments and tends to $-\infty$ if γ tends to the end-points of the segments. Hence, $\chi(\gamma)$ takes a maximum value at a certain interior point $\gamma = \gamma_*(h)$ of the segment $[\theta_{\max}-\pi, \theta_{\min}]$. Because the segment $[\theta_{\max}, \theta_{\min}+\pi]$ is obtained by shifting $[\theta_{\max}-\pi, \theta_{\min}]$ by π, it follows that $\chi(\gamma)$ takes a maximum value at the point $\gamma_*+\pi$ of the last segment. Calculation of the derivatives yields the inequality

$$\chi''(\gamma) = -2g\int\limits_0^1 \frac{H(1+2\sin^2(\theta-\gamma))\,d\lambda}{q^2\sin^4(\theta-\gamma)} < 0.$$

Then, $\chi'(\gamma) < 0$ for $\gamma \in (\gamma_*, \theta_{\min})$ and $\chi'(\gamma) > 0$ for $\gamma \in (\theta_{\max}, \gamma_*+\pi)$. From the aforesaid it follows that if $\chi(\gamma_*) = \chi(\gamma_*+\pi) > 0$, Eq. (5.4) has only two different by the modulus π roots (γ_1 and γ_2) that satisfy the inequalities $\gamma_* < \gamma_1 < \theta_{\min}$ and $\theta_{\max} < \gamma_2 < \gamma_*+\pi$. If $\chi(\gamma_*) = \chi(\gamma_*+\pi) < 0$, Eq. (5.4) has no real roots, and in the case $\chi(\gamma_*) = \chi(\gamma_*+\pi) = 0$, the root becomes multiple (by the modulus π), which leads to degeneration of systems of this type. We note that $0 < \theta-\gamma < \pi$ for $\gamma \in (\gamma_*, \theta_{\min})$ and $-\pi < \theta-\gamma < 0$ for $\gamma \in (\theta_{\max}, \gamma_*+\pi)$.

Thus, the necessary and sufficient condition of the existence of different (by the modulus π) real roots of the characteristic equation (5.4) reduces to the inequalities $\theta_{\max}(h) - \theta_{\min}(h) < \pi$ and $\chi(\gamma_*) > 0$. The latter must be satisfied only at the points γ_*, where $\chi'(\gamma_*) = 0$. It is easy to show that for flows without a vertical velocity shear (i.e., for $\theta_\lambda = 0$ and $q_\lambda = 0$), the criterion formulated reduces to the well-known condition of flow supercriticality $q > (gh)^{1/2}$.

Differentiation of the characteristic equation (5.4) with respect to the variable h yields the following integrodifferential equation for the function $\gamma(h)$:

$$\gamma_h = -\frac{3g}{2}\left(\int_0^1 \frac{H\cos(\theta-\gamma)\,d\lambda}{q^2\sin^3(\theta-\gamma)}\right)^{-1}\int_0^1 \frac{H\,d\lambda}{q^4\sin^4(\theta-\gamma)}. \tag{5.5}$$

Differentiating (5.3) with respect to λ, we obtain the following differential equations for the derivatives q_λ and θ_λ:

$$q_{\lambda h} = \frac{gq_\lambda}{q^2}, \qquad \theta_{\lambda h} = \frac{g\theta_\lambda}{q^2\sin^2(\theta-\gamma)} + \frac{2g\cot\,(\theta-\gamma)q_\lambda}{q^3}. \tag{5.6}$$

The system of integrodifferential equations (5.3), (5.5), and (5.6) can be written in the form of (4.2) for the unknown vector $\boldsymbol{V} = (q, q_\lambda, \theta, \theta_\lambda, H, \gamma)$. The simple wave joins continuously given uniform shear flow in which $h = h_0$, $q = q_0(\lambda)$, $\theta = \theta_0(\lambda)$, $H = H_0(\lambda)$, and $\gamma = \gamma_0 = \text{const}$, if appropriate Cauchy conditions $\boldsymbol{V}(h_0, \lambda) = \boldsymbol{V}_0(\lambda)$ are specified for $h = h_0$. Here γ_0 is a root of the characteristic equation (5.4) calculated for $q = q_0(\lambda)$, $\theta = \theta_0(\lambda)$, and $H = H_0(\lambda)$. In the Eulerian variables, the uniform shear flow parameters are specified by the relations $u = u_0(z)$, $v = v_0(z)$, $w = 0$, and $h = h_0 = \text{const}$.

Using the theorem of solvability of the Cauchy problem for the differential equation in Banach space, the existence of a simple wave for $|h - h_0| < \varepsilon(\delta)$ can be proved (in the same manner as in Sec. 4) if the initial data are continuous and satisfy the inequalities $H_0 > \delta > 0$, $\pi - \delta > |\theta_0 - \gamma_0| > \delta > 0$, and $|\gamma_0 - \gamma_*(h_0)| > \delta > 0$. We assume that the condition providing for the existence of different real roots of the characteristic equation is also satisfied. If a solution of Eqs. (5.3), (5.5), and (5.6) is found, the function $h(x, y)$ is determined locally in a certain subdomain of the plane (x, y) using the equation

$$\boldsymbol{n}(h)\cdot\boldsymbol{x} = m(h). \tag{5.7}$$

Here $\boldsymbol{n}(h) = (-\sin\gamma(h), \cos\gamma(h))$ and $m(h)$ is an arbitrary function.

6. Analogue of the Prandtl–Mayer Simple Wave for Shear Flows. In gas dynamics and shallow-water theory, the steady-state simple-wave solution describing flow around a convex corner — Prandtl–Mayer flow — is known [10]. Below, we extend this solution to flows with a vertical shear.

Integrating the first equation of (5.3), we obtain an analogue of the Bernoulli integral:

$$q^2 + 2gh = q_m^2(\lambda), \tag{6.1}$$

where $q_m(\lambda)$ is an arbitrary positive function. We find a particular solution of Eqs. (5.3) assuming that the modulus of the horizontal velocity is independent of the vertical Lagrangian coordinate λ. This corresponds to the following choice of the arbitrary function in (6.1): $q_m(\lambda) = \text{const}$. Then, from Eqs. (5.3) and (5.6), we have

$$(\theta_\lambda/H)_h = 0.$$

Let us consider a particular solution of this equation $\theta_\lambda/H = A$, where $A = \text{const}$. Using the relation obtained, we find

$$\theta = A\Phi + \theta_0 = Az + \theta_0,$$

where the angle $\theta_0(h)$ between the x axis and the velocity direction on the bottom satisfies the differential equation

$$\theta_{0h} = -g\cot\,(\theta_0 - \gamma)/q^2.$$

The integral in (5.4) converges for some real γ only if $\theta_{\max} - \theta_{\min} = |A|h < \pi$ in the domain of definition of the simple wave. The characteristic equation (5.4) is integrated to give

$$\chi(\gamma) = 1 + g(\cot\,(\theta_1 - \gamma) - \cot\,(\theta_0 - \gamma))/(Aq^2) = 0. \tag{6.2}$$

Here $\theta_1 = \theta_0 + Ah$. Representing $\theta_1 - \gamma$ as $\theta_1 - \theta_0 + \theta_0 - \gamma$ and using the trigonometric formula for the cotangent of the sum of the angles, we reduce Eq. (6.2) to a quadratic equation for $\cot\,(\theta_0 - \gamma)/q^2$. From this equation, we find

$$\frac{\cot\,(\theta_0 - \gamma_{1,2})}{q^2} = \frac{1}{g}\Big(\frac{A}{2} \pm \sqrt{\frac{A^2}{4} + g\,\frac{Aq^2\cot\,(\theta_1 - \theta_0) - g}{q^4}}\Big). \tag{6.3}$$

In this expression, the plus sign corresponds to the root of the characteristic equation $\gamma_1 \in (\gamma_*, \theta_{\min})$ and the minus sign corresponds to the root $\gamma_2 \in (\theta_{\max}, \gamma_* + \pi)$. Correspondingly, we consider simple waves of either the first or

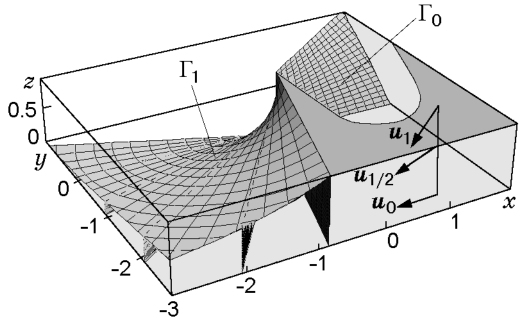

Fig. 1

second characteristic families. For the solution $\gamma_* = (\theta_1 + \theta_0)/2 - \pi/2$ considered here, we have $\theta_{\max} = \theta_1$ and $\theta_{\min} = \theta_0$ for $A > 0$ and $\theta_{\max} = \theta_0$ and $\theta_{\min} = \theta_1$ for $A < 0$. The radicand on the right side of (6.3) is positive if

$$q^2 = q_m^2 - 2gh > 2g \tan\,((\theta_1 - \theta_0)/2)/A = 2g \tan\,(Ah/2)/A. \tag{6.4}$$

For the equality in (6.4), we obtain an equation that allows us to find unambiguously the critical depth h_k. In a simple wave domain, the condition $0 < h < h_k$ must be satisfied. As $A \to 0$ (transition to shearless flow), Eq. (6.4) transforms to the classical condition of flow supercriticality $q > \sqrt{gh}$. Because $\tan x > x$ for $x \in (0, \pi/2)$, it follows from (6.4) that the condition of flow supercriticality $q > \sqrt{gh}$ holds for $A \neq 0$ as well. Hence, Eq. (6.4) is a stricter requirement (compared with the supercriticality condition). Generally, if $A \neq 0$, the functions $\theta_{0(1,2)}(h)$ are given by the quadrature

$$\theta_{0(1,2)}(h) = -\int\limits_{h_0}^{h} \left(\frac{A}{2} \pm \sqrt{\frac{A^2}{4} + g\, \frac{A(q_m^2 - 2gh')\cot\,(Ah') - g}{(q_m^2 - 2gh')^2}} \right) dh' + \theta_{00},$$

where $\theta_{00} = \text{const}$. Next, from (6.3), we find the slope of the characteristic to the x axis

$$\gamma_1(h) = \theta_{01}(h) - \operatorname{arccot}\, \frac{q_m^2 - 2gh}{g} \left(\frac{A}{2} + \sqrt{\frac{A^2}{4} + g\, \frac{A(q_m^2 - 2gh)\cot\,(Ah) - g}{(q_m^2 - 2gh)^2}} \right)$$

in a simple wave of the first family or a similar angle

$$\gamma_2(h) = \theta_{02}(h) - \operatorname{arccot}\, \frac{q_m^2 - 2gh}{g} \left(\frac{A}{2} - \sqrt{\frac{A^2}{4} + g\, \frac{A(q_m^2 - 2gh)\cot\,(Ah) - g}{(q_m^2 - 2gh)^2}} \right) + \pi$$

in a simple wave of the second family. In the limit $h \to 0$, which corresponds to a simple wave flow over a dry bottom, we have $\gamma_1(h) \to \theta_{01}(0)$ and $\gamma_2(h) \to \theta_{02}(0)$. To complete the description of simple waves in Eulerian–Lagrangian coordinates, it is necessary to integrate additionally the second equation in system (5.3) with respect to $\theta(h, \lambda)$ and calculate $H(h, \lambda) = A^{-1}\theta_\lambda(h, \lambda)$. However, in the Eulerian description, the solution is known. It is given by the relations

$$q = \sqrt{q_m^2 - 2gh}, \qquad \theta = Az + \theta_{0i}(h), \qquad \gamma_i = \gamma_i(h),$$

where $\theta_{0i}(h)$ and $\gamma_i(h)$ were calculated above. For $i = 1$ and 2, we obtain a simple wave of the first and second family, respectively.

We consider the wave centered on the line $x = 0$, $y = 0$ in the space (x, y, z). This implies that the characteristic planes $h = \text{const}$ pass through this line. Then, relation (5.7) takes the form

$$-x \sin\gamma + y \cos\gamma = 0. \tag{6.5}$$

We introduce cylindrical coordinates r, φ, and z in the space using the equalities $x = r\cos\varphi$ and $y = r\sin\varphi$. From the simplified equality (6.5) $\gamma_i(h) = \varphi$, we find the function $h = h(\varphi)$ when the function $\gamma_i(h)$ is known. These formulas define completely a centered simple wave in Eulerian coordinates. As $A \to 0$, this solution reduces to a classical Prandtl–Mayer wave.

Let us give a physical interpretation of the solution derived and describe some of its characteristics. Let a uniform shear flow

$$h = h_0, \qquad q = q_0 = \sqrt{q_m^2 - 2gh_0}, \qquad \theta = Az + \theta_{00}$$

($0 < h_0 < h_k$ and $q_m^2 = q_0^2 + 2gh_0$) move along a bank whose shape is a developable surface Γ_0 (see Fig. 1) defined by the equation

$$\varphi = Az + \theta_{00}.$$

Here the polar angle φ varies within $\theta_{00} \leqslant \varphi \leqslant Ah_0 + \theta_{00}$. The uniform shear flow adjoins a centered simple wave of the second family along the characteristic $\varphi = \gamma_2(h_0)$. In the simple-wave region $\gamma_2(h_0) > \varphi > \gamma_2(h_1)$, where $h_1 < h_0$ and the equation $\varphi = \gamma_2(h_1)$ defines the closing simple-wave characteristic, the flow is transformed into another uniform shear flow

$$h = h_1, \qquad q = q_1 = \sqrt{q_m^2 - 2gh_1}, \qquad \theta = Az + \theta_{02}(h_1),$$

which moves along an overhanging bank having the shape of a developable surface $\varphi = Az + \theta_{02}(h_1)$ for $\theta_{02}(h_1) < \varphi < Ah_0 + \theta_{02}(h_1)$. In the simple-wave region, the free surface Γ_1 defined by the equation $z = h(\varphi)$ is also developable. In the simple wave of the second family, the depth h decreases from h_0 to h_1, and the velocity modulus q increases from q_0 to q_1 with increase in the angle φ. Figure 1 shows the flow pattern in a simple centered wave during spreading of shear flow over the dry channel for $A = \pi/(5h_0)$, $\theta_{00} = 0$, and $q_0/\sqrt{gh_0} = 1.56$. The scales along the coordinate axes are referred to h_0. Figure 1 also shows the variation of the horizontal velocity component $\boldsymbol{u}$ over the depths in the incident uniform shear flow. The subscripts 0, 1/2, and 1 correspond to the values of the velocity at $z = 0$, $z = h_0/2$, and $z = h_0$, respectively. The characteristic surfaces $h = \text{const}$ and the surface Γ_0 pass through the straight line $x = 0$, $y = 0$, $0 \leqslant z \leqslant h$.

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